APPLICATION OF THE INVESTOR'S PROBLEM TO FINANCIAL MARKET OF SECURITIES

Zastavnyi N. V.
ORCID ID: 0000-0003-2097-3582
National University of "Kyiv-Mohyla Academy"

Tyshchenko S. V.
ORCID ID: 0000-0003-1804-8494
National University of "Kyiv-Mohyla Academy"

Zhukovska O. A.
Cand. physics and mathematics, associate professor
ORCID ID: 0000-0003-1110-9696
National Technical University of Ukraine “Igor Sikorsky Kyiv Polytechnic Institute”

Shchestyuk N. U.
Cand. physics and mathematics, associate professor
ORCID ID: 0000-0002-7652-8157
National University of "Kyiv-Mohyla Academy"

The article investigates the application of the investor problem to securities trading in single-period and multi-period models. The investor has to determine at stage $t$ the sum $x_t$ he invests in the purchase of securities (shares or options), if from previous historical data he knows the distribution function of a random variable $S_t$ – the share price, provided that with $E(S_t)<\infty$ in period $t$. The investor's income is a random variable $Y_{t+1}$ at stage $t$. In addition, the investor knows the bank interest rates on both deposit and loan. The relevance of this study is of particular importance in a crisis; in particular, the study of the relationship between the optimal investment in securities (in terms of profit maximization) and the management and measuring of the risk of such investment comes to the fore. This problem can be solved by the existence of a close relationship between the optimal investment and the measurement of risk given by stochastic optimization. It is possible because the average value-at-risk AV@R is related to a simple stochastic optimization problem with a piecewise linear profit/cost function and maximal value is attained. The problem includes consideration of two possible scenarios: when the income $Y_{t+1}$ at the end of the period is more than the investment $x_t$ at the beginning of the period and when the income is less than the investment and there is a shortfall. The result of the work is the formulation of four statements about the values of optimal investments in stocks and options in the single-period and multi-period models and about the potential maximum profits for such applying such a strategy. It turned out that the value of optimal investment can be directly expressed in terms of VaR of some probability level, and this level is expressed in terms of credit and deposit interest rates and characterizes the state of the national economic
environment. The results of the theoretical study were illustrated on real financial data using two models of financial markets. In the first model, the stock movement obeys the geometric Brownian motion (GBM model). The second model was a model in which the movement of risky assets is described by generalized diffusion with a new fractal time (FAT model). The results of optimal investments for both models were compared with the fair price obtained by the Black-Scholes formula. The results of the study conducted in this work were used to develop recommendations for making decisions about optimal investments in securities (stocks, options) to obtain speculative income in stock trading.

**Keywords:** investor problem, measuring risk, value at risk, risk management.
mathematical expectation, $U(\cdot)$ is a utility function, and where $X(T)$ represents the wealth at the final time $T$.

There are many works devoted to different modifications of this problem (see, e.g., Merton (1969) and survey in Hakansson (1997) and Karatzas and Shreve (1998)). In the setting generally assumed in finance, see Merton (1990), the coefficients are assumed to satisfy an Ito equation. Then the solution of the optimal investment problem can be obtained via dynamic programming approach. However, it is not easy to find the explicit solution by this method, because the corresponding Bellman equation is usually degenerate. Explicit formulas for optimal strategies have been obtained only for a few cases where appreciation rates are assumed to be non-random and known. In paper [2] the optimal investment problem was considered for a diffusion market consisting of a finite number of risky assets and stated as a problem with a maximin performance criterion. This criterion is to ensure that a strategy is found such that the minimum of utility over all distributions of parameters is maximal.

Setting objectives. The purpose of this paper is to get solution of investment problem using such probability functionals for risk measuring as \( \text{VaR} \) and \( \text{AVaR} \) because following the financial tsunami experiences of 2008 and crisis Covid19, the risk controls of risky assets and derivative instruments on stocks have become tremendously important for investors. Due to our approach, the investor gets a tool, which allows him to integrate the investment problem with risk management.

Problem statement. We consider multi-stage decision problem, which is a multi-period generalization of the simple investment problem [1].

An investor has to determine at stage $t$ the amount $x_t$ he will invest in a good opportunity at stage $t + 1$. From the regular business, he gets an income $Y_t$ with $E|Y_t| < \infty$ at time $t$.

If an investor has a speculative profit from stock trading, his investment $x_t$ at time $t$ is equal to the price of shares $S_t$, and the income from the sale of shares $Y_{t+1}$ at time $t + 1$ coincides with the value $S_{t+1}$.

In the case of options trading, its investment $x_t$ at time $t$ is equal to $C_t$ - the premium of the option (market price of the option) for a given strike price $K_t$, and the income from the realization of options $Y_{t+1}$ at the moment $t + 1$ is defined as $Y_{t+1} = [S_{t+1} - K_t]^+$ in the case of call options and $Y_{t+1} = [S_{t+1} - K_t]^-$ - in the case of put options.

If the available funds are less than the committed amount $x_t$, a shortfall occurs, which causes unit costs of $u_t > 1$. If however the funds are more than $x_t$, the surplus can be carried over to the next period, but it loses $1 - l_t$ of its value, where $0 \leq l_t \leq 1$.

If the funds $Y_{t+1}$ received from the sale of securities are less than the invested amount $x_t$, there is a shortfall, which increases according to the coefficient $u_t > 1$. If $Y_t$ exceeds $x_t$, the surplus can be carried forward, but it loses $1 - l_t$ per unit of its value, where the discount rate is $0 \leq l_t \leq 1$.

Denote by $k_t$ - (random) surplus carried over from period $t$ to period $t + 1$. And $k_0 = 0$, then:
\[ k_t = [l_{t-1}k_{t-1} + Y_t - x_{t-1}]^+, \quad t = 1, \ldots, T. \]

And let the shortfall \( m_t \):
\[ m_t = [l_{t-1}k_{t-1} + Y_t - x_{t-1}]^- . \]

Both equations can be combined into one
\[ l_{t-1}k_{t-1} + Y_t - x_{t-1} = k_t - M_t; \quad k_t \geq 0, M_t \geq 0. \]  \( \text{(1)} \)

Then the Present Value of income:
\[ H(x_0, Y_1, \ldots, x_{T-1}, Y_T) = \sum_{t=1}^{T} (x_{t-1} - u_tM_t) + l_T k_T. \]  \( \text{(2)} \)

The problem is to maximize the expected profit \( E(H(x_0, Y_1, \ldots, x_{T-1}, Y_T)) \) subject to (1).

**Methodology.** For solving this stochastic problem, we use probability functionals \( VaR, AVaR \) and their properties.

For one period model and for a given portfolio, time horizon \( T \), and probability \( p \), the \( p-VaR \) can be defined informally as the maximum possible loss during that time after we exclude all worse outcomes whose combined probability is at most \( p \). More formally, \( p-VaR \) is defined such that the probability of a loss greater than \( VaR \) is (at most) \( p \) while the probability of a loss less than \( VaR \) is (at least) \( 1 - p \). Common parameters for standard \( VaR \) are 1% and 5% probabilities and one day and two weeks horizons, although other combinations are in use.

In context of our problem the \( \alpha \)-quantile of the profit distribution \( VaR_{\alpha}(Y) = G^{-1}(\alpha), 0 < \alpha < 1 \) \( \text{(3)} \) is called value-at-risk of level \( \alpha \). Although \( VaR \) is a very popular measure of risk, it has undesirable mathematical characteristics such as a lack of subadditivity and convexity. As an alternative measure of risk, the average value-at-risk \( AVaR \) is known to have better properties than \( VaR \).

The average value-at-risk at level \( \alpha \), \( 0 < \alpha \leq 1 \) of \( Y \) is defined as
\[ AVaR_{\alpha}(Y) = \frac{1}{\alpha} \int_{0}^{\alpha} G^{-1}(u)du \]  \( \text{(4)} \)
where \( G \) is the distribution function of \( Y \).

The average value-at-risk has very useful property, it may be represented as the optimal value of the following optimization problem [1], [8]:
\[ AVaR_{\alpha}(Y) = \max \left\{ x - \frac{1}{\alpha}E([Y - x]^-) : x \in R \right\}. \]  \( \text{(5)} \)

Multi-period average value-at-risk. Let \( Y = (Y_1, \ldots, Y_T) \) be an integrable stochastic process.

For a given sequence of constants \( c = (c_1, \ldots, c_T) \), probabilities \( \alpha = (\alpha_1, \ldots, \alpha_T) \), and a filtration \( F = (F_0, \ldots, F_{T-1}) \), the multi-period average value-at-risk is defined as in [1]:
\[ AVaR_{\alpha,c}(Y; F) = \sum_{t=1}^{T} c_tE[AVaR_{\alpha_t}(Y_t|F_{t-1})]. \]

**Results of the research.** One-period model. Let \( A \) be the maximum value of the expected present value of income. If we use property \( \text{(5)} \) to the optimization problem (1)–(2) for \( t = 1 \), then we have:
\[ A = \max E(H) = (1 - l_1)AVaR_\alpha(Y_1) + l_1E(Y_1), \]

where \( \alpha = \frac{1-l}{u-l} \), \( l_1 \) is a discount factor.

In this case, the optimal amount of investment is
\[ x^* = G^{-1}(\alpha) = VaR_\alpha(G), \]

where \( G \) is the distribution function of \( Y \) [1].

Suppose we have a market, where \( S \) evolves GBM and there are two banking processes with interest rate \( R \) for deposit and \( r \) for landing. Then the following propositions are true.

**Proposition 1:** Under the 1-period model, the optimal decision for investment in share is the value \( x^* = G^{-1}(\alpha) = VaR_\alpha(G) \), where \( G \) is the lognormal distribution function for stock price movements. The maximum value of speculative income is
\[ A = (1 - l)AVaR_\alpha(G) + lE(G), \]

where \( \alpha = \frac{1-e^{-rT}}{e^{RT}-e^{-rT}}, l_1 = e^{-rT} \).

**Proposition 2:** Under the 1-period model, the optimal decision for investment in option is the value \( x^* = G^{-1}(\alpha) \), where the \( G \) is distribution function for pay off \( Y = [S_1 - K_1]^+ \) for the Call option and for \( Y = [S_1 - K_1]^− \) for the Put-option. The maximum value of speculative income:

For Call-option:
\[ A = (1 - l_1)AVaR([S_1 - K_1]^+) + l_1E([S_1 - K_1]^+), \]

for Put-option:
\[ A = (1 - l_1)AVaR([S_1 - K_1]^−) + l_1E([S_1 - K_1]^−). \]

**Multi-period model**

For \( t = 2 \):
\[ k_2 = [l_1k_1 + Y_2 - x_1]^+, \]
\[ M_2 = [l_1k_1 + Y_2 - x_1]^−. \]

Both equations can be combined into one:
\[ l_1k_1 + Y_2 - x_1 = k_2 - M_2, \quad k_2 \geq 0, \quad M_2 \geq 0. \]

For \( t = 1 \) we have:
\[ Y_1 - x_0 = k_1 - M_1 \]

Total income function for two periods:
\[ H(x_0, Y_1, x_1, Y_2) = x_0 - u_1M_1 + x_1 - u_2M_2 + l_2k_2 \]

According to the first period:
\[ H(x_0, Y_1) = x_0 - u_1M_1 + l_1k_1 = x_0 - u_1[Y_1 - x_0]^{-} + l_1[Y_1 - x_0]^{+} \]

then the total income can be written as
\[ H(x_0, Y_1, x_1, Y_2) = (x_0 - u_1M_1 + l_1k_1) + x_1 - u_2M_2 + l_2k_2 - l_1k_1. \]

Out problem is
\[ \max E(H(x_0, Y_1, x_1, Y_2)) = \]
\[ = \max E(x_0 - u_1M_1 + l_1k_1) + \max E(x_1 - u_2M_2 + l_2k_2) - l_1k_1. \]

This can be rewritten as:
\[ \max E(H(x_0, Y_1, x_1, Y_2)) = \max E(x_0, Y_1) + \max E(x_1, Y_2) - l_1E(k_1). \]
So, the maximum value of the mathematical expectation for the total period is not equal to the sum of the mathematical expectations for the individual periods. Let’s try to write this in terms of \( AVaR \)

\[
\max E(x_0, Y_1) = (1 - l_1)AVaR_{\alpha_1}(Y_1) + l_1 E(Y_1).
\]

Easy to show that

\[
\max E(x_1, Y_2) = (1 - l_2)AVaR_{\alpha_2}(Y_2) + l_2 E(Y_2).
\]

Then

\[
\max E(H(x_0, Y_1, x_1, Y_2)) =
\]

\[
= (1 - l_1)AVaR_{\alpha}(Y_1) + (1 - l_2)AVaR_{\alpha}(Y_2) + l_1 E(Y_1) + l_2 E(Y_2) - l_1 E(k_1).
\]

Now we generalize formula (7) for the case \( t = T \). Recall that \( A \) is the maximum expected present value of the entire operation, provided that the filtering \( F_{t-1} \) is the result of all previous decisions \( x_{t-1} \) at the time of period \( t - 1 \).

\[
A(Y_1, ..., Y_T, F_0, ..., F_T) = \max \{ \sum_{t=1}^{T} (x_{t-1} - u_t M_t) + l_T k_T : x_t \} \quad (8)
\]

Problem (8) has its dual representation

\[
A(Y_1, ..., Y_T, F_0) = \inf \{ \sum_{t=1}^{T} E(Y_t Z_t : Z \in Z_s) \}
\]

and can be rewritten as [1]:

\[
A(Y_1, ..., Y_T, F_0, ..., F_T) = \sum_{t=1}^{T} l_t E(Y_t) + \sum_{t=1}^{T} (1 - l_t)E[AVaR_{\alpha_t}(Y_t|F_{t-1})]
\]

Proposition 3. In a multi-period model, the optimal price (in terms of income) is the value

\[
x^*_t = VaR_{\beta_{t+1}}(Y_{t+1}|F_t) - l_t k_t.
\]

And the maximum value of speculative income for options, where

\[
Y = [S_1 - K_1]^+ \quad \text{for Call-option}
\]

and

\[
Y = [S_1 - K_1]^\pm \quad \text{for Put-option},
\]

and for Call-option

\[
\sum_{t=1}^{T} l_t E([S_1 - K_1]^+) + \sum_{t=1}^{T} (1 - l_t)E[AVaR_{\beta,C}([S_1 - K_1]^+, ..., [S_T - K_t]^+, F)],
\]

for Put-option

\[
\sum_{t=1}^{T} l_t E([S_1 - K_1]^-) + \sum_{t=1}^{T} (1 - l_t)E[AVaR_{\beta,C}([S_1 - K_1]^-, ..., [S_T - K_t]^-, F)]
\]

Remark 1. Assume that the decision maker is a clairvoyant for each period \( F_t \). Then \( x_{t-1} = Y_t \) is the optimal solution, \( M_t = 0 \) and \( k_t = 0 \) for all \( t \).

Numerical results for investment in European call options on the Apple Inc. stocks are demonstrated. We considered spot price \( S_0 = 277.0 \) for March 14, 2020. The strike price for call options with maturity \( T = 1/12 \) year is set at \( K = 255; 260; 265; 270 \), the yearly volatility for returns of the underlying asset is computed...
at $\sigma = 33.7$ percents, the yearly riskless interest rate is set at $i = 5.8\%$. To illustrate the approach we propose, we consider now the case where the yearly interest rates for borrowing is $R = 5.4$ and for lending is $r = 1.2\%$. Then $\alpha = 0.82$ due (6) for one month. For finding optimal investment we need to build empirically or theoretically distribution function for payoff and then calculate a quantile for this distribution of level $\alpha$.

On the picture 1 you can see histogram of $\vartheta_i = \frac{n_i}{n \times h}$ (where $n$ is the total amount of data in the time series; $h$ is the width of one column of the histogram), histogram of $\vartheta_i$ for $S - K$ and histogram $\vartheta_i$ for $[S - K]^+$. On the picture 2 cumulative distribution function for $[S - K]^+$ was built.

Now we can calculate a quantile for this distribution of level $\alpha = 0.82$ and the optimal solution according (6) $x^* = 32.49$. For comparing optimal solutions for different strike prices, we constructed Table 1.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$255.00$</th>
<th>$260.00$</th>
<th>$265.00$</th>
<th>$270.00$</th>
<th>$275.00$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^*$</td>
<td>$32.49$</td>
<td>$28.56$</td>
<td>$23.34$</td>
<td>$18.59$</td>
<td>$13.05$</td>
</tr>
</tbody>
</table>

**GBM model.** Sometimes it is possible to assume a known distribution function for $S$. For example, for the Black-Scholes model it is known that returns involve as GBM (Geometrical Brownian Motion) and stock prices $S$ have a lognormal distribution (see, for example [3], [4]). The lognormal density function is as follows:

$$f_X(x) = \frac{1}{x \sigma \sqrt{2\pi}} e^{-(\ln x - \mu)^2 / 2\sigma^2},$$
where $\mu$ – expectation; $\sigma$ – standard deviation.

For $\mu = 5.35$, $\sigma = 33.7\%$ we construct an integral function and calculate the quantile for $\alpha = 0.82$. Its value is $x^* = VaR_\alpha([S - K]^+) = 32.0$.

![Figure 3 – Integration distribution function and quantile for $[S - K]^+$ for GBM.](image)

Quantiles for other values of the strike price $K$ are given in Table 2.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$255.00$</th>
<th>$260.00$</th>
<th>$265.00$</th>
<th>$270.00$</th>
<th>$275.00$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^*$</td>
<td>$32.02$</td>
<td>$27.02$</td>
<td>$22.02$</td>
<td>$17.02$</td>
<td>$12.02$</td>
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**FAT model.** Consider a model for stock prices with fractal active time $T_t, t \geq 0$:

$$P_t = P_0e^{\mu t + \theta T_t + \sigma W_t}, t \geq 0,$$

where the parameters $\mu \in R$ and $\sigma > 0$ reflect the shift and volatility and $\theta \in R$.

Then if the time increments $\tau_t$ are $\text{R} \Gamma \left(\frac{\nu}{2}, \frac{\delta^2}{2}\right)$, where $\delta > 0$, $\nu > 0$, then the log returns have a Student’s distribution [5], [6]. Thus, prices have a Student’s log distribution with density

$$f_{\text{logSt}}(x) = \frac{\Gamma\left(\frac{\nu + 1}{2}\right)}{x\sigma\delta\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{1}{1 + \left(\frac{\nu}{\delta} \frac{\ln x - \mu}{\delta}\right)^{\frac{\nu + 1}{2}}}, x \in R,$$

where $\mu$ is the local parameter; $\sigma$ is the standard deviation; $\sigma$ is the scaling parameter ($\delta > 0$); $\nu$ is the number of degrees of freedom ($\nu > 0$); $\Gamma$ is a gamma function [7].

From the previous statistics we obtain the following values of parameters: $\mu = 5.35$, $\sigma = 0.34$, $\nu = 5$, $\delta = 0.8$. The graph of the probability density function is shown in Figure 4, the integral function in Figure 5. The value of the quantile for is $x^* = VaR_\alpha([S - K]^+) = 37.8$. 
Figure 4 – Density distribution function $[S - K]^+$ for FAT model

Figure 5 – Integral function distribution $[S - K]^+$ for FAT model

Quantiles for other values of the strike price $K$ are given in Table 3.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$255.00$</th>
<th>$260.00$</th>
<th>$265.00$</th>
<th>$270.00$</th>
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<tbody>
<tr>
<td>$x^*$</td>
<td>$37.77$</td>
<td>$32.77$</td>
<td>$27.77$</td>
<td>$22.77$</td>
<td>$17.77$</td>
</tr>
</tbody>
</table>

The following fig. 6 shows graphs of the optimal investment in options for different strike prices under the 1-period model. Similar results were obtained for the $n$-period model.

**Conclusions.** In the course of this work, the application of the investor problem to securities trading in single-period and multi-period models were analyzed. Numerical results for optimal investment in European call options on the Apple Inc. stocks are demonstrated.

The results of the paper were used to develop recommendations for making decisions about optimal investments in the financial market of securities.
Figure 6 – Comparing optimal investments for different models of financial markets

References: